Pergamon

Int. J. Solids Structures Vol. 32, No. 21, pp. 3087–3104, 1995

0020–7683/95 $9.50 + .00

0020–7683(94)00271–1

CLOSED-FORM SOLUTION FOR THE ISOTHERMAL CREEP RUPTURE BEHAVIOUR OF A TWO-BAR STRUCTURE UNDER CONSTANT LOAD

ORLANDO J. A. GONÇALVES FILHO
Instituto de Engenharia Nuclear CNEN, C.P. 68550, CEP 21945-970, Rio de Janeiro, RJ, Brasil

(*Received* 14 *September* 1993; *in revised form* 3 *November* 1994)

Abstract—The integro-differential equation governing the isothermal creep rupture behaviour of a kinematically determined two-bar structure under steady load is derived and solved in analytical form. One of the bars is described by an elasto-creep material model that accounts for the three phases of creep deformation and for the full coupling between the creep deformation and damage processes. The second bar is assumed to have a linear elastic behaviour. The non-linearity of the constitutive equations and the stress redistribution induced by the development of creep damage are the major physical features dealt with in this study. Numerical applications illustrate the mathematical features of the analytical solution obtained.

1. INTRODUCTION

Structural analyses of engineering components operating in the material creep range require the use of powerful numerical techniques, such as the finite element method. Analytical solutions for case studies are, nevertheless, useful on two accounts: first, by providing exact solutions against which to compare the approximate numerical results and, second, by fostering insights into the behaviour of more complex structures.

Closed-form solutions for creep deformation problems are generally restricted to transient and steady state analyses of simple metallic components. For creep rupture problems they are limited to situations where either the structure is statically determined or the elastic and primary creep strains are neglected.

This work presents a closed-form solution for the isothermal creep rupture of a two-bar structure subjected to a constant load and constrained to move in the direction of the applied load. Both bars are stressed to a low fraction of their material yield stress at the temperature considered but only one of them is assumed to undergo creep deformation. Under these loading conditions rupture of the creeping bar is essentially brittle and occurs at small strain level with no significant reduction of the bar cross-section. This type of rupture is assigned to physical damage of the material microstructure characterized by the formation, growth and coalescence of microcracks and voids at the grain boundaries. To model this low-stress isothermal creep rupture behaviour an elasto-creep theory is employed describing the primary, secondary and tertiary phases of creep deformation and the full coupling between the deformation and damage processes. The second bar is assumed to have a simple linear elastic behaviour.

In the structural model studied as damage grows stresses are transferred from the elasto-creeping bar (EC-bar) to the elastic bar (E-bar) with greater load-carrying capacity. With the collapse of the creeping bar the applied load is then sustained by the remaining elastic bar. Consequently, there is no failure in a global structural sense. This problem may thus be viewed as the brittle rupture under isothermal creep conditions of an uniaxially stressed bar that unloads with the degradation of its microstructure.

The high non-linearity of the constitutive equations and the stress redistribution process induced by the development of material damage are the physical features dealt with in this study. Together with the propagation of the failure front these features are the major sources of numerical difficulties associated with the solution of actual engineering creep rupture problems. To the author knowledge, the results reported are the first analytical solution obtained for a creep rupture problem governed by an elasto-creep theory which considers the phenomenon of stress redistribution induced by material damage. By this reason, the results derived, although referring to a relatively simple structural problem, may certainly contribute to the assessment of finite element softwares that incorporate a multiaxial generalization of the material model adopted. Numerical examples illustrate the mathematical features of the analytical solution obtained. An *ad hoc* extension of these features to multiaxial stress states is also briefly addressed.

2. PROBLEM DESCRIPTION AND MATERIAL LAWS

The two-bar structure examined is shown in Fig. 1. It consists of two parallel bars of the lengths L_1 and L_2, and cross-sectional areas S_1 and S_2. One end of each bar is pinned to a rigid abutment and the other end to a rigid block whose motion is constrained to be vertical. A downward vertical load P is applied to the rigid block.

Deformation behaviour of the first bar is described by an elasto-creep model which accounts for the primary, secondary and tertiary phases of creep deformation, and for the full coupling between the creep deformation and damage processes. According to this model, the total strain $\varepsilon_1^T(t)$ can be decomposed into elastic $\varepsilon_1^E(t)$ and creep $\varepsilon_1^C(t)$ strain parts (strain partitioning rule). The former is governed by Hooke's law while the latter is given by the extended form of Kachanov's (1958) constitutive equations proposed by Rabotnov (1969). In rate form these constitutive equations are written as

$$\frac{d\varepsilon_1^T(t)}{dt} = \frac{d\varepsilon_1^E(t)}{dt} + \frac{d\varepsilon_1^C(t)}{dt} \tag{1}$$

$$\frac{d\varepsilon_1^E(t)}{dt} = \frac{1}{E_1}\frac{d\sigma_1(t)}{dt} \tag{2}$$

$$\frac{d\varepsilon_1^C(t)}{dt} = B_1\left(\frac{\sigma_1(t)}{1-\omega_1(t)}\right)^{N_1} t^{m_1}, \tag{3}$$

where E_1 is the Young's modulus, $\sigma_1(t)$ the nominal stress, B_1, N_1 and $m_1(-1 < m_1 \leqslant 0)$ material properties and $\omega_1(t)$ the scalar variable describing the current state of material damage. In the initial undamaged state ω_1 is equal to zero and during the deformation process the value of ω_1 increases monotonically from zero to a critical value ω_{cr} (close to unity) at final failure.

The evolution of the damage variable is given by

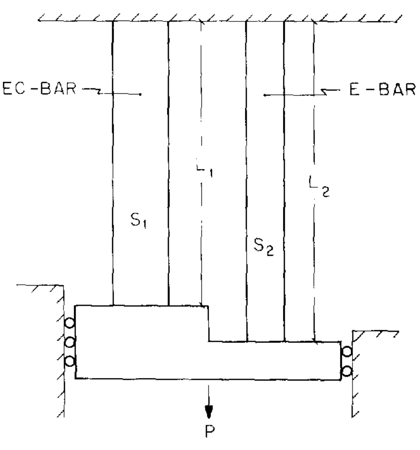

Fig. 1. Two-bar geometry.

$$\frac{\mathrm{d}\omega_1(t)}{\mathrm{d}t} = A_1 \frac{[\sigma_1(t)]^{\chi_1}}{[1-\omega_1(t)]^{\phi_1}} t^{m_1}; \quad (0 \leqslant \omega_1(t) \leqslant \omega_{cr}), \tag{4}$$

where A_1, χ_1 and ϕ_1 are additional material properties. The polynomial term in the time variable t^{m_1} has been included in the creep strain rate and damage rate equations (3) and (4) to account for primary creep effects.

In his original work Kachanov (1958) interpreted the damage variable as the fraction of the cross-sectional area of an element of volume which is occupied by either voids or internal fissures. The net stress acting over the effective cross-section of an uniaxially tensile specimen is, therefore, $\sigma/(1-\omega)$, where σ is the nominal applied stress. Clearly the net stress is greater than the nominal stress and increases considerably at final rupture. Rabotnov (1969) pointed out, however, that such physical interpretation was not strictly necessary in the context of the phenomenological approach proposed and this viewpoint is followed here. Kachanov–Rabotnov's equations were later generalized to multiaxial stress states by Leckie and Hayhurst (1974) (isotropic creep damage theory) and by Murakami and Ohno (1981) (anisotropic creep damage theory), among others.

The second bar is assumed to have a simple linear elastic behaviour governed by Hooke's law, so that

$$\frac{\mathrm{d}\varepsilon_2^T(t)}{\mathrm{d}t} = \frac{\mathrm{d}\varepsilon_2^E(t)}{\mathrm{d}t} = \frac{1}{E_2}\frac{\mathrm{d}\sigma_2(t)}{\mathrm{d}t}. \tag{5}$$

Before deriving the problem governing equation it is convenient to introduce a new time scale t^*, given by

$$t^* = \int_0^t \tau^{m_1}\,\mathrm{d}\tau = t^{m_1+1} \tag{6}$$

in order to eliminate the explicit dependence of time in the creep strain and damage evolution equations (3) and (4). Considering eqn (6), the constitutive equations (1) to (5) are then rewritten as

$$\dot{\varepsilon}_1^T(t^*) = \dot{\varepsilon}_1^E(t^*) + \dot{\varepsilon}_1^C(t^*) \tag{7}$$

$$\dot{\varepsilon}_1^E(t^*) = \frac{1}{E_1}\dot{\sigma}_1(t^*) \tag{8}$$

$$\dot{\varepsilon}_1^C(t^*) = \frac{B_1}{m_1+1}\left(\frac{\sigma_1(t^*)}{1-\omega_1(t^*)}\right)^{\lambda_1} \tag{9}$$

$$\dot{\omega}_1(t^*) = \frac{A_1}{m_1+1}\frac{[\sigma_1(t^*)]^{\chi_1}}{[1-\omega_1(t^*)]^{\phi_1}} \tag{10}$$

$$\dot{\varepsilon}_2^T(t^*) = \dot{\varepsilon}_2^E(t^*) = \frac{1}{E_2}\dot{\sigma}_2(t^*), \tag{11}$$

where the overdot indicates differentiation with respect to the new time scale t^*.

3. GOVERNING INTEGRO-DIFFERENTIAL EQUATION

Equilibrium of forces in the vertical direction requires that at any instant

$$\sigma_1(t^*)S_1 + \sigma_2(t^*)S_2 = P. \tag{12}$$

Compatibility of displacements in turn imposes

$$\varepsilon_1^{\mathrm{T}}(t^*)L_1 = \varepsilon_2^{\mathrm{T}}(t^*)L_2 = \Delta, \tag{13}$$

where Δ is the vertical displacement of the rigid moveable block.

At the instant of loading ($t^* = 0$) the instantaneous elastic response is

$$\sigma_i(0) = \left(\frac{k_i}{k_1 + k_2}\right)\frac{P}{S_i} \tag{14}$$

and

$$\varepsilon_i^{\mathrm{T}}(0) = \varepsilon_i^{\mathrm{E}}(0) = \frac{\sigma_i(0)}{E_i}, \tag{15}$$

where $k_i = S_i E_i / L_i$ is the elastic stiffness of the i-th bar ($i = 1, 2$).

Substitution of eqns (7)–(9) and (11) into eqn (13) in rate form yields

$$\dot{\varepsilon}_1^{\mathrm{T}}(t^*)L_1 = \left\{\frac{\dot{\sigma}_1(t^*)}{E_1} + \frac{B_1}{m_1+1}\left[\frac{\sigma_1(t^*)}{1-\omega_1(t^*)}\right]^{N_1}\right\}L_1 = \frac{\dot{\sigma}_2(t^*)}{E_2}L_2. \tag{16}$$

Rearranging terms and considering eqn (12) in rate form, it is possible to rewrite eqn (16) as

$$\frac{\dot{\sigma}_1(t^*)}{E^*} + \frac{B_1}{m_1+1}\left(\frac{\sigma_1(t^*)}{1-\omega_1(t^*)}\right)^{N_1} = 0, \tag{17}$$

where $E^* = [k_2/(k_1+k_2)]E_1$.

The damage evolution eqn (10) can be integrated to give

$$\omega_1(t^*) = 1 - \left\{1 - \frac{A_1(1+\phi_1)}{m_1+1}\int_0^{t^*}[\sigma_1(\tau)]^{\chi_1}\,\mathrm{d}\tau\right\}^{\frac{1}{1+\phi_1}}. \tag{18}$$

Substitution of eqn (18) into eqn (17) leads finally to

$$\frac{\dot{\sigma}_1(t^*)}{E^*} + \frac{B_1}{m_1+1}\,\frac{[\sigma_1(t^*)]^{N_1}}{\left\{1 - \frac{A_1(1+\phi_1)}{m_1+1}\int_0^{t^*}[\sigma_1(\tau)]^{\chi_1}\,\mathrm{d}\tau\right\}^{\frac{N_1}{1+\phi_1}}} = 0, \tag{19}$$

which is the nonlinear integro-differential equation in $\sigma_1(t^*)$ governing the deformation and rupture behaviour of the two-bar structure under examination. Initial conditions are given by the elastic response (eqns 14 and 15).

4. PROBLEM SOLUTION

Let

$$y(t^*) = \left\{1 - \frac{A_1(1+\phi_1)}{m_1+1}\int_0^{t^*} [\sigma_1(\tau)]^{\chi_1}\, \mathrm{d}\tau\right\}^{\frac{N_1}{1+\phi_1}}. \tag{20}$$

Differentiating eqn (20) with respect to time yields

$$\dot{y}(t^*) = -\left(\frac{N_1 A_1}{m_1+1}\right)[y(t^*)]^{1-\left(\frac{1+\phi_1}{N_1}\right)}[\sigma_1(t^*)]^{\chi_1}, \tag{21}$$

from which

$$\sigma_1(t^*) = \left\{\frac{-\dot{y}(t^*)}{\left(\frac{N_1 A_1}{m_1+1}\right)[y(t^*)]^{1-\left(\frac{1+\phi_1}{N_1}\right)}}\right\}^{\frac{1}{\chi_1}}. \tag{22}$$

At the initial time ($t^* = 0$), eqns (14), (20) and (21) give

$$y(0) = 1 \tag{23}$$

and

$$\dot{y}(0) = -\left(\frac{N_1 A_1}{m_1+1}\right)[\sigma_1(0)]^{\chi_1} = -\left(\frac{N_1 A_1}{m_1+1}\right)\left[\left(\frac{k_1}{k_1+k_2}\right)\frac{P}{S_1}\right]^{\chi_1}. \tag{24}$$

Differentiating eqn (21) with respect to time and introducing eqn (22) results in

$$\ddot{y}(t^*) = \frac{\dot{y}(t^*)}{y(t^*)}\left\{\left[1-\left(\frac{1+\phi_1}{N_1}\right)\right]\dot{y}(t^*) + \frac{\left(\frac{N_1 A_1}{m_1+1}\right)^{\frac{1}{\chi_1}}\chi_1}{[-\dot{y}(t^*)]^{\frac{1}{\chi_1}}}[y(t^*)]^{1+\left[1-\left(\frac{1+\phi_1}{N_1}\right)\right]\frac{1}{\chi_1}}\dot{\sigma}_1(t^*)\right\} \tag{25}$$

from which

$$\dot{\sigma}_1(t^*) = \frac{-\ddot{y}(t^*) + \left[1-\left(\frac{1+\phi_1}{N_1}\right)\right]\frac{[\dot{y}(t^*)]^2}{y(t^*)}}{\left(\frac{N_1 A_1}{m_1+1}\right)^{\frac{1}{\chi_1}}\chi_1[-\dot{y}(t^*)]^{1-\frac{1}{\chi_1}}[y(t^*)]^{\left[1-\left(\frac{1+\phi_1}{N_1}\right)\right]\frac{1}{\chi_1}}}. \tag{26}$$

Substituting eqns (20), (22) and (26) into eqn (19) yields

$$\mathscr{A} y(t^*)\ddot{y}(t^*) - \mathscr{A}\mathscr{C}[\dot{y}(t^*)]^2 - \mathscr{B}[-\dot{y}(t^*)]^{1-\mathscr{G}}[y(t^*)]^{\mathscr{C}\mathscr{G}} = 0, \tag{27}$$

where

$$\mathscr{A} = \frac{1}{E^*\left(\frac{N_1 A_1}{m_1+1}\right)^{\frac{1}{\chi_1}}\chi_1}; \quad \mathscr{B} = \frac{B_1}{(m_1+1)\left(\frac{N_1 A_1}{m_1+1}\right)^{\frac{N_1}{\chi_1}}} \tag{28}$$

$$\mathscr{C} = 1-\left(\frac{1+\phi_1}{N_1}\right); \qquad \mathscr{D} = \frac{1-N_1}{\chi_1}.$$

Equation (27) is the nonlinear second-order ordinary differential equation corresponding to the original integro-differential eqn (19).

For solution of eqn (27) the following change of variables is introduced

$$p(y) = -\dot{y}(t^*) \tag{29}$$

leading to

$$\mathscr{A}y(t^*)p(y)\frac{\mathrm{d}p(y)}{\mathrm{d}y} - \mathscr{A}\mathscr{C}[p(y)]^2 - \mathscr{B}[p(y)]^{1-\mathscr{D}}[y(t^*)]^{\mathscr{C}\mathscr{D}} = 0, \tag{30}$$

or

$$\frac{\mathrm{d}p(y)}{\mathrm{d}y} + \left(-\mathscr{C}\frac{1}{y(t^*)}\right)p(y) = \left(\frac{\mathscr{B}}{\mathscr{A}}[y(t^*)]^{\mathscr{C}\mathscr{D}-1}\right)[p(y)]^{-\mathscr{D}}, \tag{31}$$

which is a classical nonlinear differential equation of first-order in $p(y)$ with variable coefficients known as Bernoulli's equation.

A Bernoulli's type equation can be solved by the method of separable variables. Introducing

$$p(y) = u(y)v(y) \tag{32}$$

eqn (31) yields

$$u'(y)v(y) + u(y)\left[v'(y) - \frac{\mathscr{C}}{y(t^*)}v(y)\right] = \frac{\mathscr{B}}{\mathscr{A}}[y(t^*)]^{\mathscr{C}\mathscr{D}-1}[v(y)]^{-\mathscr{D}}[u(y)]^{-\mathscr{D}}, \tag{33}$$

where the superscript prime indicates differentiation with respect to $y(t^*)$.

The function $v(y)$ is determined as one particular solution which cancels the expression between parentheses in the second term of the left-hand side of eqn (33), e.g.

$$v'(y) - \frac{\mathscr{C}}{y(t^*)}v(y) = 0. \rightarrow v(y) = [y(t^*)]^{\mathscr{C}}. \tag{34}$$

Substituting the expression for $v(y)$ back into eqn (33) yields

$$u'(y) = \frac{\mathscr{B}}{\mathscr{A}}y^{-\mathscr{C}-1}[u(y)]^{-\mathscr{D}} \tag{35}$$

from which

$$\int [u(y)]^{\mathscr{D}}\, \mathrm{d}u = \frac{\mathscr{B}}{\mathscr{A}} \int y^{-\mathscr{C}-1}\, \mathrm{d}y. \tag{36}$$

It will be shown in the following that an analytical solution for the $u(y)$ function in eqn (36) leading to a closed-form solution for the integro-differential problem of interest is possible with

$$\mathscr{D} = -1 \quad \text{and} \quad \mathscr{C} = 0. \tag{37}$$

It is not difficult to verify that other combinations for $\mathscr{D}$ and $\mathscr{C}$ lead to recurrence integration formulae for evaluation of the $u(y)$ function rendering it almost impossible to obtain closed-form solutions for the original problem of interest.

Conditions (37) imply that (see eqn 28)

$$\chi_1 = \phi_1 = N_1 - 1(>0) \tag{38}$$

and also that

$$\frac{\mathscr{B}}{\mathscr{A}} = \frac{(N_1-1)E^*B_1}{N_1A_1}. \tag{39}$$

Equation (36) subjected to the constraints imposed by eqn (37) simplifies to

$$\int \frac{\mathrm{d}u}{u(y)} = \frac{(N_1-1)E^*B_1}{N_1A_1} \int \frac{\mathrm{d}y}{y}. \tag{40}$$

Integration of eqn (40) yields

$$u(y) = K_1 y^{\frac{(N_1-1)E^*B_1}{N_1A_1}}, \tag{41}$$

where K_1 is a constant to be determined.

With $v(y)$ and $u(y)$ functions determined, $p(y)$ is then obtained as (eqn 32)

$$p(y) = u(y) \quad v(y) = K_1 y^{\frac{(N_1-1)E^*B_1}{N_1A_1}} \quad y^0 = K_1 y^{\frac{(N_1-1)E^*B_1}{N_1A_1}}. \tag{42}$$

The unknown function $y(t^*)$ is now determined by considering eqns (29) and (42). Thus from

$$-\frac{\mathrm{d}y}{p(y)} = \mathrm{d}t^* \tag{43}$$

it follows that

$$-\int_{y(0)=1}^{y(t^*)} y^{-\frac{(N_1-1)E^*B_1}{N_1A_1}}\, \mathrm{d}y = \int_0^{t^*} K_1\, \mathrm{d}\tau. \tag{44}$$

Two cases must now be considered:

Case (1): $(N_1-1)E^*B_1 \neq N_1A_1$

From eqn (44) it follows that

$$y(t^*) = \left[1-\left(\frac{N_1A_1-(N_1-1)E^*B_1}{N_1A_1}\right)K_1t^*\right]^{\frac{N_1A_1}{N_1A_1-(N_1-1)E^*B_1}}. \tag{45}$$

Differentiating eqn (45) with respect to time yields

$$\dot{y}(t^*) = -K_1\left[1-\left(\frac{N_1A_1-(N_1-1)E^*B_1}{N_1A_1}\right)K_1t^*\right]^{\frac{(N_1-1)E^*B_1}{N_1A_1-(N_1-1)E^*B_1}}. \tag{46}$$

Constant K_1 is obtained considering the initial condition (24) and eqns (38) and (46) as

$$K_1 = \left(\frac{N_1A_1}{m_1+1}\right)\left[\left(\frac{k_1}{k_1+k_2}\right)\frac{P}{S_1}\right]^{N_1-1}. \tag{47}$$

The stress history for the EC-bar can now be determined from eqn (22) by considering eqns (38) and (45)–(47) as

$$\sigma_1(t^*) = \sigma_1(0)\left\{1-\left(\frac{N_1A_1-(N_1-1)E^*B_1}{m_1+1}\right)[\sigma_1(0)]^{N_1-1}t^*\right\}^{\frac{E^*B_1}{N_1A_1-(N_1-1)E^*B_1}}. \tag{48}$$

The stress history for the E-bar in turn is obtained from eqns (12) and (48) as

$$\sigma_2(t^*) = \frac{P}{S_2}-\frac{S_1}{S_2}\sigma_1(t^*). \tag{49}$$

The damage growth in the EC-bar is obtained from eqns (18), (38) and (48) as

$$\omega_1(t^*) = 1-\left[1-\left(\frac{N_1A_1-(N_1-1)E^*B_1}{m_1+1}\right)[\sigma_1(0)]^{N_1-1}t^*\right]^{\frac{A_1}{N_1A_1-(N_1-1)E^*B_1}}. \tag{50}$$

By recalling that at rupture $\omega_1(t_R^*) = \omega_{cr}$, the EC-bar lifetime is determined as

$$t_R^* = \left(\frac{1}{\sigma_1(0)}\right)^{N_1-1}\left(\frac{m_1+1}{N_1A_1-(N_1-1)E^*B_1}\right)\left[1-(1-\omega_{cr})^{\frac{N_1A_1-(N_1-1)E^*B_1}{A_1}}\right], \tag{51}$$

where $0 < \omega_{cr} < 1.0$.

The total strain history of the EC-bar is determined from the elastic and creep strain histories according to the strain partitioning rule (eqn (7)). The former is determined immediately by considering eqn (8) in non-rate form and eqn (48). The creep strain history in turn is obtained by integrating the creep strain rate (eqn (9)) after properly introducing eqns (48) and (50). Thus,

$$\varepsilon_1^{\mathrm{T}}(t^*) = \frac{\sigma_1(0)}{E_1}\left\{\left[1-\left(\frac{N_1A_1-(N_1-1)E^*B_1}{m_1+1}\right)[\sigma_1(0)]^{N_1-1}t^*\right]^{\frac{E^*B_1}{N_1A_1-(N_1-1)E^*B_1}}\right.$$

$$\left.-\frac{E_1}{E^*}\left[\left(1-\left(\frac{N_1A_1-(N_1-1)E^*B_1}{m_1+1}\right)[\sigma_1(0)]^{N_1-1}t^*\right)^{\frac{E^*B_1}{N_1A_1-(N_1-1)E^*B_1}}-1\right]\right\}. \quad (52)$$

The total strain of the E-bar is easily obtained from eqns (13) and (52) as

$$\varepsilon_2^{\mathrm{T}}(t^*) = \varepsilon_2^{\mathrm{E}}(t^*) = \frac{L_1}{L_2}\varepsilon_1^{\mathrm{T}}(t^*). \quad (53)$$

Case (2) : $(N_1-1)E^*B_1 = N_1A_1$.
From eqn (44) it now follows that

$$y(t^*) = \mathrm{e}^{-K_1t^*}. \quad (54)$$

Differentiating eqn (54) with respect to time yields

$$\dot{y}(t^*) = -K_1\,\mathrm{e}^{-K_1t^*}. \quad (55)$$

Constant K_1 is obtained considering the initial condition (24) and eqns (38) and (55) as

$$K_1 = \left(\frac{N_1A_1}{m_1+1}\right)\left[\left(\frac{k_1}{k_1+k_2}\right)\frac{P}{S_1}\right]^{N_1-1}. \quad (56)$$

The stress history for the EC-bar can now be determined from eqn (22) by considering eqns (38) and (54) to (56) as

$$\sigma_1(t^*) = \sigma_1(0)\,\mathrm{e}^{\frac{(-N_1A_1)}{(N_1-1)(m_1+1)}[\sigma_1(0)]^{N_1-1}t^*}. \quad (57)$$

The stress history for the E-bar in turn is obtained from eqns (12) and (57) as

$$\sigma_2(t^*) = \frac{P}{S_2}-\frac{S_1}{S_2}\sigma_1(t^*). \quad (58)$$

The damage growth in the EC-bar is obtained from eqns (18), (38) and (57) as

$$\omega_1(t^*) = 1-\mathrm{e}^{\left(\frac{-A_1}{m_1+1}\right)[\sigma_1(0)]^{N_1-1}t^*}. \quad (59)$$

By recalling that at rupture $\omega_1(t_{\mathrm{R}}^*) = \omega_{\mathrm{cr}}$, the EC-bar lifetime is determined as

$$t_{\mathrm{R}}^* = \left(\frac{1}{\sigma_1(0)}\right)^{N_1-1}\left(\frac{m_1+1}{A_1}\right)\ln\left(\frac{1}{1-\omega_{\mathrm{cr}}}\right), \quad (60)$$

where $0 < \omega_{\mathrm{cr}} < 1.0$.

The total strain history of the EC-bar is determined from the elastic and creep strain histories according to the strain partitioning rule (eqn 7). The former is determined immediately by considering eqn (8) in non-rate form and eqn (57). The creep strain history in turn is obtained by integrating the creep strain rate (eqn 9) after properly considering eqns (57), (59) and the condition $(N_1-1)E^*B_1 = N_1A_1$. Thus.

$$\varepsilon_1^{\mathrm{T}}(t^*) = \frac{\sigma_1(0)}{E_1}\left\{\mathrm{e}^{\frac{(-N_1A_1)}{(N_1-1)(m_1+1)}[\sigma_1(0)]^{N_1-1}t^*} - \frac{E_1}{E^*}\left[\mathrm{e}^{\frac{(-N_1A_1)}{(N_1-1)(m_1+1)}[\sigma_1(0)]^{N_1-1}t^*} - 1\right]\right\}. \tag{61}$$

The total strain of the E-bar is obtained from eqns (13) and (61) as

$$\varepsilon_2^{\mathrm{T}}(t^*) = \varepsilon_2^{\mathrm{E}}(t^*) = \frac{L_1}{L_2}\varepsilon_1^{\mathrm{T}}(t^*). \tag{62}$$

To obtain the results in terms of the real time scale t, it remains to consider the inverse of the transformation eqn (6). For ease of reference a summary of the results thus derived is compiled next in normalized form.

5. FORMULA SUMMARY

Case (1): $(N_1-1)E^*B_1 \neq N_1A_1$: $(0 < \omega_{\mathrm{cr}} < 1)$.

Stress histories:

$$\frac{\sigma_1(t/t_{\mathrm{R}})}{\sigma_1(0)} = \left\{1-\left[1-(1-\omega_{\mathrm{cr}})^{\frac{N_1A_1-(N_1-1)E^*B_1}{A_1}}\right]\left(\frac{t}{t_{\mathrm{R}}}\right)^{m_1+1}\right\}^{\frac{E^*B_1}{N_1A_1-(N_1-1)E^*B_1}},$$

$$\frac{\sigma_2(t/t_{\mathrm{R}})}{\sigma_2(0)} = \frac{E_1}{E^*} + \left(1-\frac{E_1}{E^*}\right)\frac{\sigma_1(t/t_{\mathrm{R}})}{\sigma_1(0)}, \quad \text{where}$$

$$\sigma_i(0) = \left(\frac{k_i}{k_1+k_2}\right)\frac{P}{S_i}, \quad k_i = \frac{S_iE_i}{L_i}(i=1,2) \quad \text{and} \quad E^* = \left(\frac{k_2}{k_1+k_2}\right)E_1.$$

Total strain histories:

$$\frac{\varepsilon_1^{\mathrm{T}}(t/t_{\mathrm{R}})}{\varepsilon_1^{\mathrm{T}}(0)} = \frac{\varepsilon_2^{\mathrm{T}}(t/t_{\mathrm{R}})}{\varepsilon_2^{\mathrm{T}}(0)} = \left\{1-\left[1-(1-\omega_{\mathrm{cr}})^{\frac{N_1A_1-(N_1-1)E^*B_1}{A_1}}\right]\left(\frac{t}{t_{\mathrm{R}}}\right)^{m_1-1}\right\}^{\frac{E^*B_1}{N_1A_1-(N_1-1)E^*B_1}}$$

$$-\frac{E_1}{E^*}\left\{\left[1-\left[1-(1-\omega_{\mathrm{cr}})^{\frac{N_1A_1-(N_1-1)E^*B_1}{A_1}}\right]\left(\frac{t}{t_{\mathrm{R}}}\right)^{m_1+1}\right]^{\frac{E^*B_1}{N_1A_1-(N_1-1)E^*B_1}}-1\right\},$$

$$\text{with} \quad \varepsilon_1^{\mathrm{T}}(0) = \frac{\sigma_1(0)}{E_1} \quad \text{and} \quad \varepsilon_2^{\mathrm{T}}(0) = \frac{\sigma_2(0)}{E_2} = \frac{L_1}{L_2}\frac{\sigma_1(0)}{E_1}.$$

Damage history:

$$\omega_1\left(\frac{t}{t_{\mathrm{R}}}\right) = 1-\left\{1-\left[1-(1-\omega_{\mathrm{cr}})^{\frac{N_1A_1-(N_1-1)E^*B_1}{A_1}}\right]\left(\frac{t}{t_{\mathrm{R}}}\right)^{m_1+1}\right\}^{\frac{A_1}{N_1A_1-(N_1-1)E^*B_1}}.$$

EC-bar lifetime:

$$t_R = \left\{\left(\frac{1}{\sigma_1(0)}\right)^{N_1-1}\left(\frac{m_1+1}{N_1A_1-(N_1-1)E^*B_1}\right)\left[1-(1-\omega_{cr})^{\frac{N_1A_1-(N_1-1)E^*B_1}{A_1}}\right]\right\}^{\frac{1}{m_1+1}}.$$

Stresses and strains at EC-bar failure ($t/t_R = 1$):

$$\frac{\sigma_1(1)}{\sigma_1(0)} = (1-\omega_{cr})^{\frac{E^*B_1}{A_1}}; \quad \frac{\sigma_2(1)}{\sigma_2(0)} = \frac{E_1}{E^*}+\left(1-\frac{E_1}{E^*}\right)(1-\omega_{cr})^{\frac{E^*B_1}{A_1}},$$

$$\frac{\varepsilon_1^T(1)}{\varepsilon_1^T(0)} = \frac{\varepsilon_2^T(1)}{\varepsilon_2^T(0)} = \frac{E_1}{E^*}+\left(1-\frac{E_1}{E^*}\right)(1-\omega_{cr})^{\frac{E^*B_1}{A_1}}.$$

Case (2): $(N_1-1)E^*B_1 = N_1A_1$; $(0 < \omega_{cr} < 1)$.

Stress histories:

$$\frac{\sigma_1(t/t_R)}{\sigma_1(0)} = (1-\omega_{cr})^{\left(\frac{N_1}{N_1-1}\right)\left(\frac{t}{t_R}\right)^{m_1+1}},$$

$$\frac{\sigma_2(t/t_R)}{\sigma_2(0)} = \frac{E_1}{E^*}+\left(1-\frac{E_1}{E^*}\right)\frac{\sigma_1(t/t_R)}{\sigma_1(0)}.$$

Total strain histories:

$$\frac{\varepsilon_1^T(t/t_R)}{\varepsilon_1^T(0)} = \frac{\varepsilon_2^T(t/t_R)}{\varepsilon_2^T(0)} = (1-\omega_{cr})^{\left(\frac{N_1}{N_1-1}\right)\left(\frac{t}{t_R}\right)^{m_1+1}} - \frac{E_1}{E^*}\left[(1-\omega_{cr})^{\left(\frac{N_1}{N_1-1}\right)\left(\frac{t}{t_R}\right)^{m_1+1}}-1\right].$$

Damage history:

$$\omega_1(t/t_R) = 1-(1-\omega_{cr})^{\left(\frac{t}{t_R}\right)^{m_1+1}}.$$

EC-bar lifetime:

$$t_R = \left[\left(\frac{1}{\sigma_1(0)}\right)^{N_1-1}\left(\frac{m_1+1}{A_1}\right)\ln\left(\frac{1}{1-\omega_{cr}}\right)\right]^{\frac{1}{m_1+1}}.$$

Stresses and strains at EC-bar failure ($t/t_R = 1$):

$$\frac{\sigma_1(1)}{\sigma_1(0)} = (1-\omega_{cr})^{\frac{N_1}{N_1-1}}; \quad \frac{\sigma_2(1)}{\sigma_2(0)} = \frac{E_1}{E^*}+\left(1-\frac{E_1}{E^*}\right)(1-\omega_{cr})^{\frac{N_1}{N_1-1}},$$

$$\frac{\varepsilon_1^T(1)}{\varepsilon_1^T(0)} = \frac{\varepsilon_2^T(1)}{\varepsilon_2^T(0)} = \frac{E_1}{E^*}+\left(1-\frac{E_1}{E^*}\right)(1-\omega_{cr})^{\frac{N_1}{N_1-1}}.$$

OBS: (1) The structural response of the EC-bar when carrying the applied load alone can be obtained by evaluating the limit of Case (1) expressions when $k_2 \to 0$. Thus,

Stress history:

$$\frac{\sigma_1(t/t_R)}{\sigma_1(0)} = 1; \quad \sigma_1(0) = \frac{P}{S_1}.$$

Total strain history:

$$\frac{\varepsilon_1^T(t/t_R)}{\varepsilon_1^T(0)} = 1 - \frac{E_1 B_1}{N_1 A_1} \ln \left\{1 - [1-(1-\omega_{cr})^{N_1}]\left(\frac{t}{t_R}\right)^{m_1+1}\right\}; \quad \varepsilon_1^T(0) = \frac{P}{E_1 S_1}.$$

Damage history:

$$\omega_1(t/t_R) = 1 - \left\{1 - [1-(1-\omega_{cr})^{N_1}]\left(\frac{t}{t_R}\right)^{m_1+1}\right\}^{\frac{1}{N_1}}.$$

EC-bar lifetime:

$$t_R = \left\{\left(\frac{1}{\sigma_1(0)}\right)^{N_1-1}\left(\frac{m_1+1}{N_1 A_1}\right)[1-(1-\omega_{cr})^{N_1}]\right\}^{\frac{1}{m_1+1}}.$$

Stress and strain at EC-bar failure ($t/t_R = 1$):

$$\frac{\sigma_1(1)}{\sigma_1(0)} = 1,$$

$$\frac{\varepsilon_1^T(1)}{\varepsilon_1^T(0)} = 1 - \frac{E_1 B_1}{A_1}\ln(1-\omega_{cr}).$$

OBS: (2) For both cases $(N_1-1)E^*B_1 \neq N_1 A_1$ and $(N_1-1)E^*B_1 = N_1 A_1$,

$$\text{when} \quad t > t_R \quad \text{then} \quad \begin{cases} \sigma_2(t/t_R) = P/S_2 \\ \varepsilon_2^T(t/t_R) = P/(E_2 S_2) \end{cases}.$$

6. NUMERICAL APPLICATION

Numerical applications are now considered to illustrate the mathematical features of the analytical solution derived. Reporting to the formula summary of Section 5 some remarks are initially in order. First it is noted that *given a specific elasto-creep material whose properties satisfy the constraints* $\chi_1 = \phi_1 = N_1 - 1$ (eqn (38)), the response of the two-bar structure to an arbitrary constant mechanical load under isothermal condition is a function of the ratio between the elastic stiffness of each bar only (through the parameter E^*). Second, when primary creep is neglected ($m_1 = 0$) and the elastic stiffness ratio (k_1/k_2) is equal to $[(E_1 B_1/A_1)-1]$, damage grows at a constant rate of ω_{cr}. This latter remark indicates that the response characteristics of the two-bar structure are dependent of the rate of damage growth, as will be clarified next.

The material selected for the elasto-creeping bar is a Ti–6Al–2Cr–2Mo titanium alloy whose properties are specified in Table 1 (Walczak *et al.*, 1983) for the temperature of 675 K. The threshold value ω_{cr} is set equal to 0.9.

The geometry of the bars and the Young's modulus of the elastic bar are selected to give specific elastic stiffness ratios, but are otherwise arbitrary. To illustrate the different

Table 1. Material properties for Ti–6Al–2Cr–2Mo titanium alloy at 675 K (Walczak *et al.*, 1983)

Property	Value
E_1	0.102×10^6 MPa
N_1	6.8
B_1	1.38×10^{-24} MPa $^{-6.8}$ h^{-1}
$\chi_1 = \phi_1$	$5.79 \cong 5.8\ (= N_1 - 1)$
A_1	1.08×10^{-20} MPa $^{-5.8}$ h^{-1}
m_1	0.0

responses of the two-bar structure, two elastic stiffness ratios are considered: (1) $(k_1/k_2) = 18$ $(>[(E_1B_1/A_1)-1])$ and (2) $(k_1/k_2) = 11$ $(<[(E_1B_1/A_1)-1])$. It is not difficult to verify that when primary creep is neglected the response pattern associated with the particular condition $(N_1-1)E^*B_1 = N_1A_1$ is similar to that corresponding to $(k_1/k_2) < [(E_1B_1/A_1)-1]$.

All results (stress, strain and damage histories) are presented in normalized form. If a specific load is required, it is recalled that the formulae derived are valid only if the elasto-creeping bar is loaded to a low fraction of its material virgin yield stress at the temperature considered (830 MPa, in this case). Specific numerical expressions for all the curves drawn are compiled in the appendix.

Figures 2 to 4 show the stress, strain and damage histories for the two-bar structure with an elastic stiffness ratio $(k_1/k_2) = 18$. For comparison, the strain and damage histories of the EC-bar when carrying the applied load alone are also presented in Figs 3 and 4, respectively. It is recalled that the (constant) stress acting on the EC-bar working alone is *greater* than the initial stress supported under the two-bar configuration. The one-bar results were further normalized with respect to the rupture time of the two-bar structure (see appendix).

The beneficial effect of the stress redistribution process on the elasto-creeping bar lifetime is clear from Figs 3 and 4; in the two-bar configuration the EC-bar lifetime increased 230%, approximately. The strain and damage histories, nevertheless, exhibit the same general pattern of evolution in time.

Complete results (stress, strain and damage histories) for both elastic stiffness ratios considered are shown together in Figs 5 to 7. For a better comparison, the results relative to the elastic stiffness ratio $(k_1/k_2) = 11$ are normalized with respect to the rupture time of the two-bar structure of ratio $(k_1/k_2) = 18$ (see appendix).

With regard to Figs 5 to 7, two broad conclusions can be drawn. First, damage grows at an *increasing* rate when $(k_1/k_2) > [(E_1B_1/A_1)-1]$ (a response similar to that observed when the creeping bar carries the applied load alone) and in a *decreasing* rate when $(k_1/k_2) < [(E_1B_1/A_1)-1]$. Second, the two-bar lifetime is markedly greater in the decreasing damage rate case. Both statements are easily confirmed by direct examination of the formulae derived.

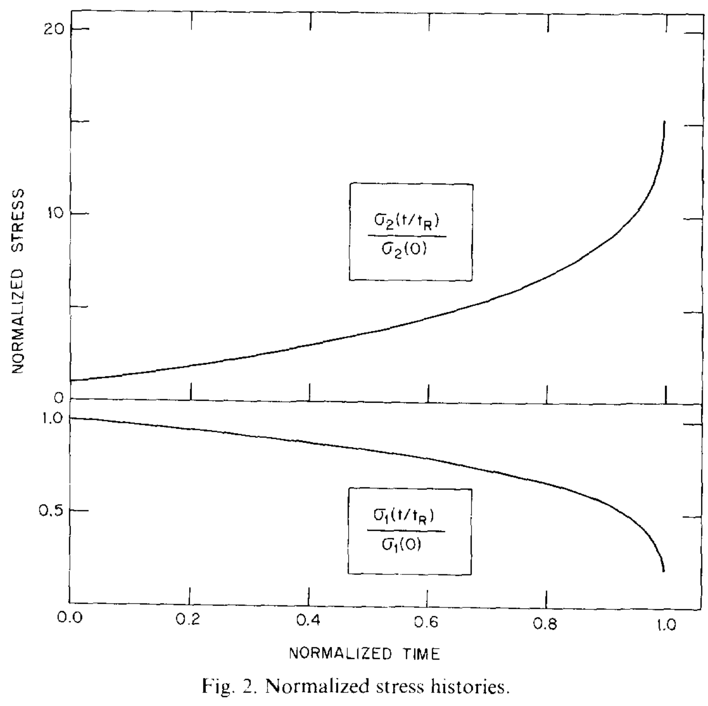

Fig. 2. Normalized stress histories.

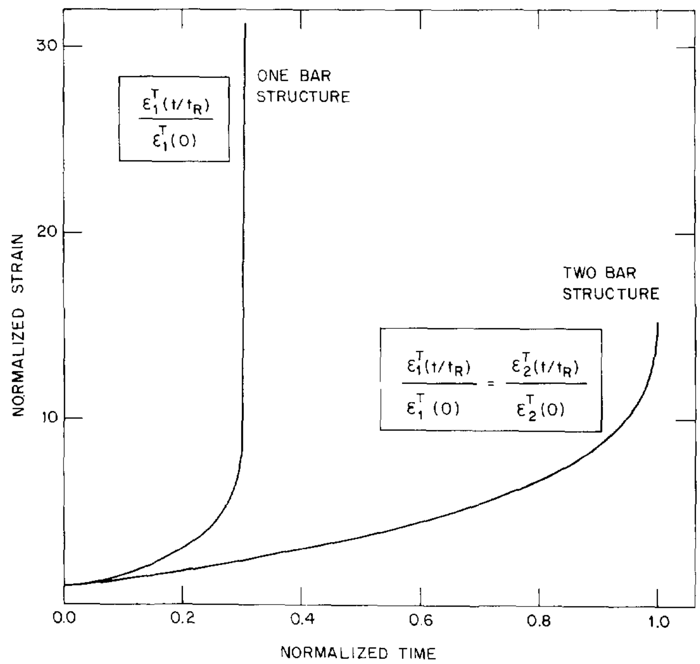

Fig. 3. Normalized total strain histories.

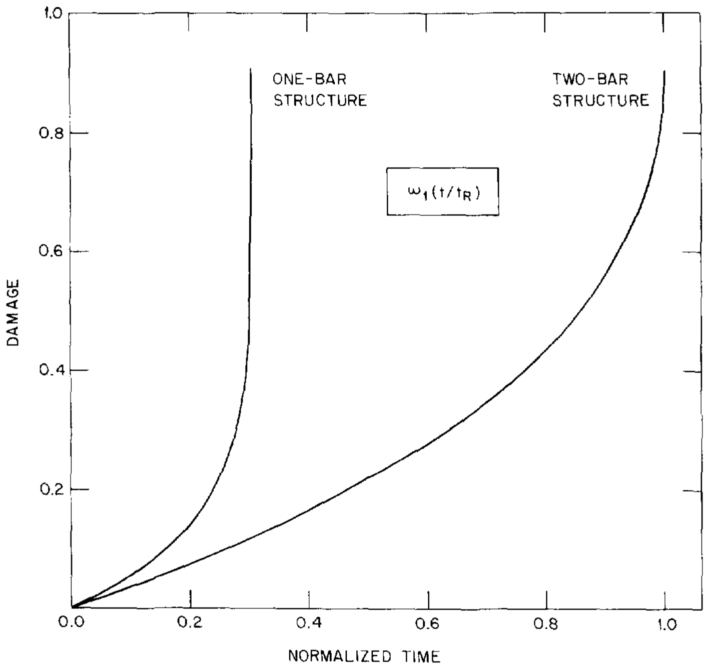

Fig. 4. Damage histories.

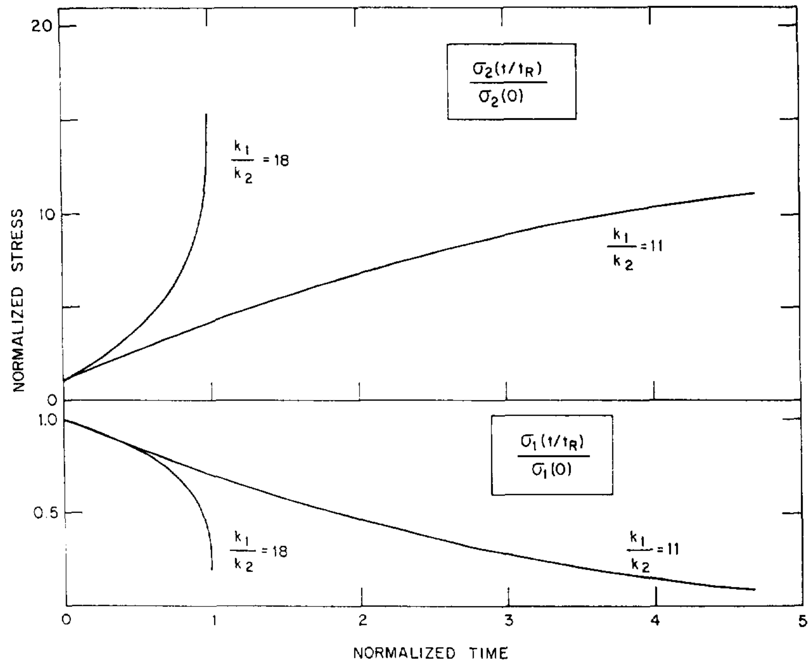

Fig. 5. Normalized stress histories.

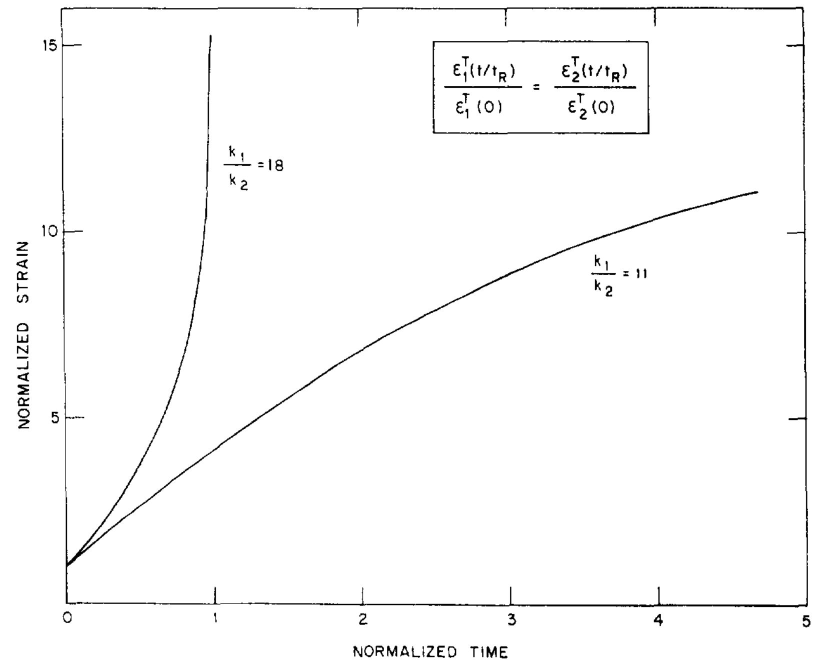

Fig. 6. Normalized total strain histories.

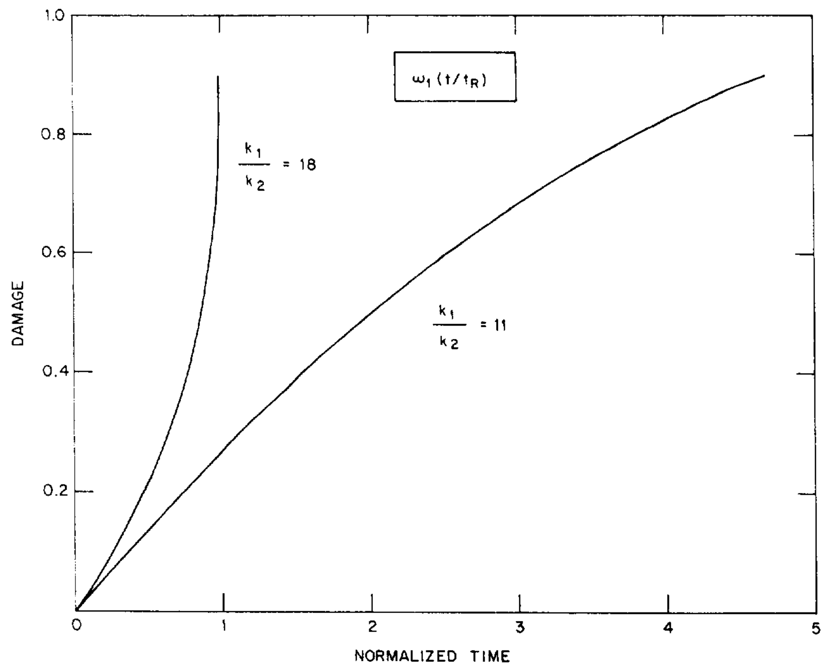

Fig. 7. Damage histories.

7. CONCLUDING REMARKS

In this work an analytical solution has been determined for the isothermal creep rupture behaviour of a two-bar structure subjected to a constant load. This problem may be viewed, essentially, as the brittle rupture under isothermal creep conditions of an uniaxially stressed bar that unloads with the degradation of its material microstructure. Although referring to a relatively simple structural problem, the solution reported, by taking into account the stress redistribution process induced by material damage, shall be useful in assessing approximate numerical results obtained through finite element softwares that incorporate a multiaxial generalization of the material model employed.

The observation that damage growth is affected by the ratio between the elastic stiffness of the creeping (damageable) and elastic (damage free) parts of the two-bar structure may be extended on an *ad hoc* basis to multiaxial stress cases and eventually exploited in high temperature design. Essentially in situations where creep deformation is expected to be confined to a homogeneously stressed zone in the component, the neighbouring elastic regions should be designed so that the ratio between the elastic stiffness of the creeping and elastic regions be lower than $[(E_1 B_1/A_1)-1]$. Hence the creeping region will develop a high fraction of damage early in a comparatively much longer life. Since regions with higher damage values are more easily detectable by non-destructive examinations, corrective measures can be taken at an early stage of the component's life, thereby considerably diminishing the risk of a sudden local collapse.

The *ad hoc* extension suggested above disregards primary creep, is restricted to materials whose properties satisfy the constraints given by eqn (38) and implicitly assumes that creep damage in multiaxial stress states can still be represented by a scalar variable. The more limiting condition, however, appears to be that of the creeping region be *homogeneously* stressed. The same stress condition is not strictly required in the neighbouring elastic region, because a non-homogeneous stress field in this region may be approximately represented by a set of (n) homogeneously stressed bars, which is a direct generalization of the problem examined. The full validation of the *ad hoc* extension proposed and its further extension to the more general case of non-homogeneous multiaxial stress states in the creep region clearly deserves additional investigation.

Acknowledgement—This work is published by permission of the Comissão Nacional de Energia Nuclear, Brazil.

REFERENCES

Kachanov, L. M. (1958). Time of the fracture process under creep conditions. *Izv. Akad. Nauk., USSR, Otd. Teck. Nauk.* **8,** 26–31.
Rabotnov, Yu. M. (1969). *Creep Problems in Structural Members.* North-Holland, Amsterdam.
Leckie, F. A. and Hayhurst, D. R. (1974). Creep rupture of structures. *Proc. R. Soc. London* **A340,** 323–347.
Murakami S. and Ohno, N. (1981). A continuum theory of creep and creep damage. In *Creep in Structures* (Edited by A. R. S. Ponter and D. R. Hayhurst). Springer, Berlin.
Walczak, J., Sieniawski, J. and Bathe, K. J. (1983). On the analysis of creep stability and rupture. *Comp. Struct.* **17,** 783–792.

APPENDIX: SPECIFIC NUMERICAL EXPRESSIONS FOR THE EXAMPLES CONSIDERED

Case (1): $(k_1/k_2) = 18(>[(E_1B_1/A_1)-1])$.

Stress histories:

$$\frac{\sigma_1(t/t_R)}{\sigma_1(0)} = \left(1-0.9985\frac{t}{t_R}\right)^{0.2431}, \qquad \sigma_1(0) = \frac{18}{19}\frac{P}{S_1},$$

$$\frac{\sigma_2(t/t_R)}{\sigma_2(0)} = 19-18\left(1-0.9985\frac{t}{t_R}\right)^{0.2431}, \quad \sigma_2(0) = \frac{1}{19}\frac{P}{S_2}.$$

Total strain histories:

$$\frac{\varepsilon_1^T(t/t_R)}{\varepsilon_1^T(0)} = \frac{\varepsilon_2^T(t/t_R)}{\varepsilon_2^T(0)} = 19-18\left(1-0.9985\frac{t}{t_R}\right)^{0.2431},$$

$$\varepsilon_1^T(0) = \frac{18}{19}\frac{P}{E_1S_1}, \quad \varepsilon_2^T(0) = \frac{1}{19}\frac{P}{E_2S_2}.$$

Damage history:

$$\omega_1(t/t_R) = 1-\left(1-0.9985\frac{t}{t_R}\right)^{0.3544}.$$

EC-bar lifetime:

$$t_R = 3.277\times 10^{19}[\sigma_1(0)]^{-5.8}.$$

Stresses and strains at EC-bar failure $(t/t_R = 1)$:

$$\frac{\sigma_1(1)}{\sigma_1(0)} = 0.205, \quad \frac{\sigma_2(1)}{\sigma_2(0)} = 15.29,$$

$$\frac{\varepsilon_1^T(1)}{\varepsilon_1^T(0)} = \frac{\varepsilon_2^T(1)}{\varepsilon_2^T(0)} = 15.29.$$

Case (2): $(k_1/k_2) = 11\ (<[E_1B_1/A_1)-1])$.

Stress histories:

$$\frac{\sigma_1(t/t_R)}{\sigma_1(0)} = \left(1-0.6842\frac{t}{t_R}\right)^{2.170}, \qquad \sigma_1(0) = \frac{11}{12}\frac{P}{S_1},$$

$$\frac{\sigma_2(t/t_R)}{\sigma_2(0)} = 12-11\left(1-0.6842\frac{t}{t_R}\right)^{2.170}, \quad \sigma_2(0) = \frac{1}{12}\frac{P}{S_2}.$$

Total strain histories:

$$\frac{\varepsilon_1^T(t/t_R)}{\varepsilon_1^T(0)} = \frac{\varepsilon_2^T(t/t_R)}{\varepsilon_2^T(0)} = 12-11\left(1-0.6842\frac{t}{t_R}\right)^{2.170}.$$

Damage history:

$$\omega_1(t/t_R) = 1-\left(1-0.6842\frac{t}{t_R}\right)^{1.998}.$$

EC-bar lifetime:

$$t_R = 1.266\times10^{20}[\sigma_1(0)]^{-5.8}.$$

Stresses and strains at EC-bar failure ($t/t_R = 1$):

$$\frac{\sigma_1(1)}{\sigma_1(0)} = 0.0820,\quad \frac{\sigma_2(1)}{\sigma_2(0)} = 11.10,$$

$$\frac{\varepsilon_1^T(1)}{\varepsilon_1^T(0)} = \frac{\varepsilon_2^T(1)}{\varepsilon_2^T(0)} = 11.10.$$

OBS: to normalize previous expressions with respect to the rupture time of Case (1),

$$\text{i.e.}\quad t_{R|(k_1/k_2)=18},\quad \text{replace}\left(\frac{t}{t_R}\right)\quad\text{by}\quad \alpha\left(\frac{t}{t_{R|(k_1/k_2)=18}}\right),\quad\text{where}$$

$$\alpha = \frac{t_{R|(k_1/k_2)=18}}{t_{R|(k_1/k_2)=11}} = \frac{3.277\times10^{19}(18/19)^{-5.8}(P/S_1)^{-5.8}}{1.266\times10^{20}(11/12)^{-5.8}(P/S_1)^{-5.8}} = 0.2138.$$

Case (3). One-bar structure (limit of Case (1) expressions when $k_2\to0$).

Stress history:

$$\frac{\sigma_1(t/t_R)}{\sigma_1(0)} = 1,\quad \sigma_1(0) = \frac{P}{S_1}.$$

Total strain history:

$$\frac{\varepsilon_1^T(t/t_R)}{\varepsilon_1^T(0)} = 1-1.917\ln\left[1-(1-0.1^{6.8})\frac{t}{t_R}\right],\quad \varepsilon_1^T(0) = \frac{P}{E_1S_1}.$$

Damage history:

$$\omega_1(t/t_R) = 1-\left[1-(1-0.1^{6.8})\frac{t}{t_R}\right]^{0.1471}.$$

EC-bar lifetime:

$$t_R = 1.362\times10^{19}[\sigma_1(0)]^{-5.8}.$$

Stress and strain at EC-bar failure ($t/t_R = 1$):

$$\frac{\sigma_1(1)}{\sigma_1(0)} = 1;\quad \frac{\varepsilon_1^T(1)}{\varepsilon_1^T(0)} = 31.01.$$

OBS: to normalize previous expressions with respect to the rupture time of Case (1),

$$\text{i.e.}\quad t_{R|(k_1/k_2)=18},\quad \text{replace}\left(\frac{t}{t_R}\right)\quad\text{by}\quad \beta\left(\frac{t}{t_{R|(k_1/k_2)=18}}\right),\quad\text{where}$$

$$\beta = \frac{t_{R|(k_1/k_2)=18}}{t_{R|(k_2=0)}} = \frac{3.277\times10^{19}(18/19)^{-5.8}(P/S_1)^{-5.8}}{1.362\times10^{19}(P/S_1)^{-5.8}} = 3.292.$$